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Spin-type model for interacting agents with memory

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This paper presents a model and a solution method to investigate the emergence of self-organization in populations where interaction between members is combined with individual memory of previous behavior. The Markov model is second order to accommodate an extended memory range including the previous as well as the next-to-previous time instant. The proposed solution method leads to a set of recurrence relations between the present and previous macrostates of the system and the equilibrium distribution of the population is then obtained as the stable fixed points of these recurrences. The approach applies to a category of interaction mechanisms and is illustrated here on a simple example showing how memory allocation may affect the emergence of a phase transition in the collective behavior of the population.

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I. INTRODUCTION

Self-organization of populations is expected to emerge as a result of elementary interactions between individuals. Often, the agents are memoryless and the evolution of the system to the next state is exclusively governed by interactions between members: past history then plays no explicit role in the self-organization process. The purpose of the present paper is to provide a model and a method to analyze how self-organization is affected when the usual interaction between agents is combined with individual interference of each agent with its own past. The impact of the past is introduced via a memory urging each agent to repeat its earlier behavior. In a first attempt along this line [1], memory was limited to the immediately preceding instant and, moreover, a simplified model was used specifying directly the probability of entire time trajectories of the population. Here, in contrast, a two-step memory range is considered, including also the next-to-previous instant, and the system is now modeled by a Markov chain, which seems more realistic since it stays closer to the step by step evolution of the population. The model and the accompanying method of resolution presented here are to some extent generic in that they apply to a category of interaction mechanisms specified below. To illustrate how they work in practice, they will be applied in Sec. III to a classical example.

Section II presents the second-order Markov chain model combining the interaction between agents with an extended memory component, prompting each individual to reproduce its behavior over the two preceding instants. For large populations, this Markov model induces a recurrence relation between past and present collective behavior and the equilibrium distribution is then given by the stable fixed points of the recurrence. The model and its recurrence solution are illustrated in Sec. III by a simple situation where the interaction between agents consists in tandem recruitment.

II. THE MARKOV MODEL AND EQUILIBRIUM DISTRIBUTION

Consider a population where the agents can be in one of two different states, labeled A and B. Let n be the size of the population and x its fraction in state A. The state of each individual j is denoted by a binary variable s_j^t indicating in which state it is at time t, with $s_j^t=1$ for state A and $s_j^t=-1$ for state B. The state or configuration of a generation at time t is given at the microscopic level by the binary vector $s^t=[s_1^t,s_2^t,\ldots,s_n^t]$ and at the macroscopic or collective level by the fraction x_t of that generation in state A with

$$x_t=\frac{1}{2}+\frac{1}{2n}\sum_{j=1}^{n} s_j^t. \tag{1}$$

The evolution of the population results from a combination of two distinct mechanisms: interaction between agents of the same generation and an interaction along the time axis of each individual with its own past via an elementary form of memory. The first of these interactions (i.e., that between agents) is represented by an energy function $E(x)$ depending on the fraction x of the population in state A. For the time being, no further information about this energy function is needed to explain the principle of the approach developed below. Application of the technique to a specific problem requires, of course, the detailed expression of $E(x)$, as will be given in the example of Sec. III.

To this interaction within the same generation, we add the effect of an individual memory extending over two time steps. For given configurations at the previous and next-to-previous instants s^t and s^{t-1}, the evolution of the population to its next configuration s^{t+1} is modeled by the second-order Markov chain

$$P(s^{t+1}|s^t,s^{t-1})=\exp\left(-E(x_{t+1})+\beta_1\sum_{j=1}^{n} s_j^{t+1}s_j^t +\beta_2\sum_{j=1}^{n} s_j^{t+1}s_j^{t-1}\right)\Big/ Z(s^t,s^{t-1}), \tag{2}$$

where normalization is realized by the partition function

$$Z(s^t,s^{t-1})=\sum_{\{s_j^{t+1}=\pm 1\}}\exp\left(-E(x_{t+1})+\beta_1\sum_{j=1}^{n} s_j^{t+1}s_j^{t}\right.$$
$$\left.+\beta_2\sum_{j=1}^{n} s_j^{t+1}s_j^{t-1}\right). \tag{3}$$

Interaction within a generation is accounted for by the factor $\exp[(-E(x_{t+1})]$ which represents the fitness of the new generation s^{t+1} with respect to this mechanism. The factors $\exp(\beta_1\Sigma s_j^{t+1}s_j^t)$ and $\exp(\beta_2\Sigma s_j^{t+1}s_j^{t-1})$ represent the conservative behavior induced by memory since their effect is to give higher probability to the next generation s^{t+1}, where the individuals show greater coherence with their choices at the previous and next-to-previous instants. The relative impact of this conservative attitude is represented by the positive memory weights β_1 and β_2.

The central question is, of course, to find the equilibrium distribution reached by the Markov model (2) as $t\to\infty$. Let us briefly outline the method to obtain this.

Closer examination of the partition function (3) will reveal that, in the thermodynamic limit of large population size $n\to\infty$, the transition probability (2) has a dominating contribution to the effect that, for given configurations at time t and $t-1$, it produces a next generation with a *deterministic* value of the fraction x_{t+1} in state A. As expected, this fraction x_{t+1} depends only on the fractions x_t,x_{t-1} of the two previous generations s^t, s^{t-1} and on their correlation $y_{t,t-1}$. It will turn out that these quantities are indeed related by an equation of the form

$$\phi(x_{t+1},x_t,x_{t-1},y_{t,t-1})=0, \tag{4}$$

from which the value of x_{t+1} can be computed. In order then to complement the macrostate description at time $t+1$, all we need is an updating equation for the correlation $y_{t+1,t}$, which in fact will take the form

$$\psi(y_{t+1,t},x_{t+1},x_t,x_{t-1},y_{t,t-1})=0. \tag{5}$$

The equilibrium values of fraction and correlation reached by the Markov chain (2) are then the stable fixed points of the second-order nonlinear recurrences (4) and (5). This general outline of the approach will now be worked out in detail.

Although by itself the partition function (3) is of no importance here, we shall nevertheless apply the techniques that are routinely used for its evaluation because, in so doing, we shall find what we are looking for, i.e., the deterministic values of fraction and correlation where the mass of the transition probability (2) is concentrated.

In order to calculate the trace in the right hand side of Eq. (3), the exponent is linearized with respect to the s_j^{t+1} via a Dirac transformation [2,3]. As a result, the partition function can be rewritten as

$$Z(s^t,s^{t-1})=(2\pi i)^{-1}\int_0^1\int_{\alpha-i\infty}^{\alpha+i\infty}$$
$$\times\exp[-nf(x_{t+1},\zeta,\beta_1,\beta_2)]dx_{t+1},d\zeta, \tag{6}$$

where

$$f(x_{t+1},\zeta,\beta_1,\beta_2)=\frac{1}{n}E(x_{t+1})-\frac{\zeta}{2n}(1-2x_{t+1})$$
$$-\frac{1}{n}\sum_{j=1}^{n}\ln\left[2\cosh\left(\frac{\zeta}{2\mathrm{n}}+\beta_1\mathrm{s}_\mathrm{j}^\mathrm{t}+\beta_2\mathrm{s}_\mathrm{j}^{\mathrm{t}-1}\right)\right]. \tag{7}$$

For large population size $(n\to\infty)$, we can apply steepest descent integration [4,5] to Eq. (6), which yields the saddle point equations defining the fraction x_{t+1} in state A where the probability mass of Eq. (2) is concentrated,

$$\zeta=-E'(x_{t+1}), \tag{8}$$

$$(2x_{t+1}-1)=\frac{1}{n}\sum_{j=1}^{n}\tanh\left(\frac{\zeta}{2n}+\beta_1 s_j^t+\beta_2 s_j^{t-1}\right). \tag{9}$$

From Eq. (3), one also recognizes that the correlation $y_{t+1,t}$ with the previous generation, i.e., the average of $(1/n)\Sigma_{j=1}^n s_j^{t+1}s_j^t$ across all configurations $\{s_j^{t+1}\}$, can be computed as $y_{t+1,t}=(1/n)(\partial/\partial\beta_1)\ln Z(s^t,s^{t-1})$. In view of Eqs. (6) and (7) this yields

$$y_{t+1,t}=\frac{1}{n}\sum_{j=1}^{n}\tanh\left(\frac{\zeta}{2n}+\beta_1 s_j^t+\beta_2 s_j^{t-1}\right)s_j^t. \tag{10}$$

The frequency of occurrence of the four sign patterns in (s_j^t,s_j^{t-1}) can easily be expressed in terms of the fractions x_t,x_{t-1} and the correlation $y_{t,t-1}$ of the generations s^t and s^{t-1}. Hence, Eqs. (9) and (10) can be rewritten as

$$4X_{t+1}=(X_t+X_{t-1}+1+y_{t,t-1})\frac{z+B_\Sigma}{1+zB_\Sigma}-(X_t+X_{t-1}-1$$
$$-y_{t,t-1})\frac{z-B_\Sigma}{1-zB_\Sigma}+(X_t-X_{t-1}+1-y_{t,t-1})\frac{z+B_\Delta}{1+zB_\Delta}$$
$$-(X_t-X_{t-1}-1+y_{t,t-1})\frac{z-B_\Delta}{1-zB_\Delta}, \tag{11}$$

$$4y_{t+1,t}=(X_t+X_{t-1}+1+y_{t,t-1})\frac{z+B_\Sigma}{1+zB_\Sigma}+(X_t+X_{t-1}-1$$
$$-y_{t,t-1})\frac{z-B_\Sigma}{1-zB_\Sigma}+(X_t-X_{t-1}+1-y_{t,t-1})\frac{z+B_\Delta}{1+zB_\Delta}$$
$$+(X_t-X_{t-1}-1+y_{t,t-1})\frac{z-B_\Delta}{1-zB_\Delta}, \tag{12}$$

with

$$B_\Sigma=\tanh(\beta_1+\beta_2),\quad B_\Delta=\tanh(\beta_1-\beta_2),\quad z=\tanh\left(\frac{\zeta}{2n}\right), \tag{13}$$

and where we have introduced the change of variable

$$X_{t+1}=(2x_{t+1}-1),\quad |X_{t+1}|\leqslant 1 \tag{14}$$

suggested by the left hand side of Eq. (9). Elimination of z in Eqs. (11), (12) via Eqs. (8) and (13) yields the recurrence relations Eqs. (4), (5). Putting there $x_{t+1}=x_t=x_{t-1}=x$ and $y_{t+1,t}=y_{t,t-1}=y$, one obtains the fixed point equations providing the macroscopic observables of the states at equilibrium. Stability of the fixed points can be verified by a first-order perturbation analysis.

III. APPLICATION TO TANDEM RECRUITMENT

Let us now illustrate the approach presented so far by applying it to the case where the interaction energy $E(x)$ between agents of the same generation is the Hamiltonian of the well known tandem recruitment in an ant colony foraging at two equivalent sources (see, e.g., [6] and the references therein). At each time step, an individual is selected, and it can switch to the opposite source, either spontaneously with a probability ϵ (inconstancy) or because it is recruited with probability $(1-\delta)$ (persuasiveness) when meeting a fellow companion feeding there. As such, tandem recruitment leads to an equilibrium distribution at the microscopic level of configurations $\rho_s=\exp[-\mathcal{H}(x)]/Z$, where the Hamiltonian is given by

$$\mathcal{H}(x)=-n[(\eta+x)\ln(\eta+x)+(\eta+1-x)\ln(\eta+1-x)] \tag{15}$$

with $\eta=\epsilon/(1-\delta)$ [1,7]. Since the number of configurations having the same fraction x is $\binom{n}{nx}$, the equilibrium distribution at the macroscopic level of the fractions is $\rho_x=\binom{n}{nx}\rho_s$. For large n, the latter distribution has a narrow peak at $x=1/2$, meaning that, when tandem recruitment is the only interaction, large populations show equal concentrations at the sources, whatever the values of inconstancy ϵ and persuasiveness $(1-\delta)$.

Taking now expression (15) as the energy function $E(x)$ in the Markov chain (2), it turns out that Eq. (8) becomes

$$X_{t+1}=H\tanh(\zeta/2n) \tag{16}$$

with

$$H=1+2\eta. \tag{17}$$

Using this equation to eliminate $z=\tanh(\zeta/2n)$ from Eqs. (11) and (12) and putting

$$\xi_t=X_t/H \tag{18}$$

yields the relevant recurrence relations (4),(5) for the case considered here.

For simplicity, let us first examine in detail the case $\beta_2=0$ where memory is restricted to the immediate past. Putting then

$$B=B_\Sigma=B_\Delta=\tanh\beta_1, \tag{19}$$

the recurrence relation (4) becomes

$$HB^2\xi_{t+1}^3-HB\xi_{t+1}^2\xi_t+(1-H-B^2)\xi_{t+1}+HB\xi_t=0. \tag{20}$$

As could be expected, a first fixed point is located at $\xi^*=0$ which means, in view of Eqs. (18) and (14), $x^*=\frac{1}{2}$, i.e., equal repartition of the population over the two food sources. A first-order perturbation analysis of Eq. (20) around this fixed point shows that it is stable if $B<H-1$. Beyond the critical value

$$\tilde{B}=H-1, \tag{21}$$

a phase transition sets in and the recurrence equation (20) has then two stable symmetrical fixed points

$$\hat{\xi}_\pm=\pm\sqrt{\frac{1+B-H}{HB}}, \tag{22}$$

which corresponds to a selective preference for one or the other of the sources, $\hat{x}_+=(1+H\hat{\xi}_+)/2>\frac{1}{2}$ and $\hat{x}_-=(1+H\hat{\xi}_-)/2<\frac{1}{2}$, depending on the initial conditions in which the population was started. In view of the definitions (17),(19) of H and B, the equal repartition between sources is maintained if $\tanh\beta_1<2\epsilon/(1-\delta)$, i.e., as long as the memory weight β_1 is low compared to the ratio of inconstancy ϵ over persuasiveness $(1-\delta)$. The memory threshold beyond which symmetry is broken becomes smaller when inconstancy is low and persuasiveness is high.

A similar discussion can be given for the two-step memory case $\beta_2>0$, which includes also the effect of correlation with the next-to-previous generation s^{t-1} in the Markov chain (2). Straightforward calculations show that the recurrence equations (11),(12) lead to an odd polynomial fixed point equation of degree 9 in ξ. The equal repartition fixed point solution $\xi^*=0$ exists and is stable as long as B_Σ stays below the critical value

$$\tilde{B}_\Sigma=\frac{(H-1)(2+B_\Delta)+B_\Delta^2}{(H+B_\Delta+1)}>0. \tag{23}$$

Beyond this value, a phase transition develops where the fixed point equation has two solutions, corresponding again to a selective concentration of the population at one or the other of the two sources. For fixed value of H, representing the effect of recruitment, the critical relation (23) is represented by a curve in the B_Σ,B_Δ plane (see Fig. 1), its relevant portion lying between the straight lines $B_\Delta=\pm B_\Sigma$ and $B_\Sigma=1$. The equal repartition solution $(\xi^*=0)$ is stable in the region to the left of the curve and becomes unstable when $B_\Sigma>\tilde{B}_\Sigma$, where the phase transition sets in and selective concentration appears. Figure 1 displays a set of such curves for different values of H: beyond $H=4$, no phase transition is possible, whatever the values of the memory parameters B_Σ and B_Δ.

The sum $\beta_1+\beta_2$, or equivalently $B_\Sigma=\tanh(\beta_1+\beta_2)$, can be considered as a measure of the total memory weight attributed to the past. For given H, Fig. 1 shows that the threshold value $\tilde{B}_\Sigma$ for a phase transition has a minimum (marked by a circle) with coordinates

$$\tilde{B}_{\Sigma\,\min}=4\sqrt{H}-(H+3), \quad \tilde{B}_{\Delta\,\min}=\sqrt{4H}-(H+1). \tag{24}$$

The value of $\tilde{B}_{\Sigma\,\min}$ gives the minimum total memory weight $\beta_1+\beta_2$ required to trigger the phase transition. Interestingly, since $\tilde{B}_{\Delta\,\min}$ is clearly negative, β_2 is then larger than β_1,

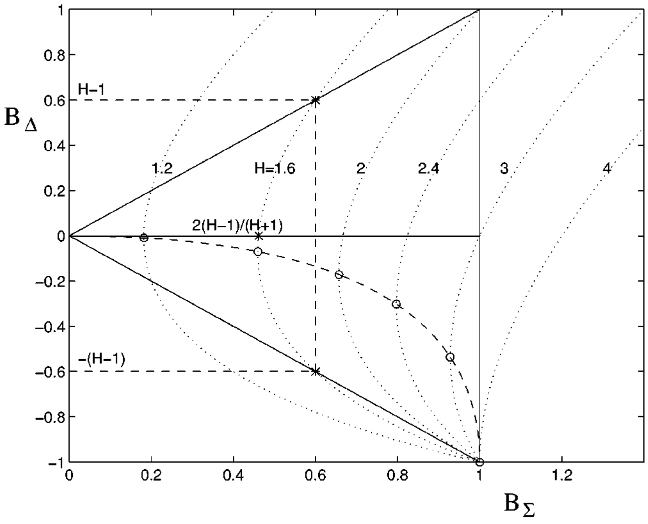

FIG. 1. The dotted lines show the critical relation (23) between system parameters $H=1+2\epsilon/(1-\delta)$, $B_\Sigma=\tanh(\beta_1+\beta_2)$, and $B_\Delta=\tanh(\beta_1-\beta_2)$. The dashed line is the locus of $\tilde{B}_{\Sigma\ \min}$, $\tilde{B}_{\Delta\ \min}$.

meaning that a phase transition at this point requires a stronger memory weight for the distant (s^{t-1}) than for the recent past (s^t). The locus of $\tilde{B}_{\Sigma\ \min}$, $\tilde{B}_{\Delta\ \min}$ drawn in dashed lines shows that $\tilde{B}_{\Delta\ \min}$ becomes more and more negative as H increases. In other words, large values of H, i.e., high inconstancy ϵ and small persuasiveness $(1-\delta)$ in the recruitment process, require that more emphasis should be put on the correlation with the next-to-previous state s^{t-1} in order to achieve phase transition with the minimum total memory weight $\beta_1+\beta_2$.

TABLE I. Comparison between the values predicted by the theory and the equilibrium fraction $\langle x\rangle$ of the population at source A averaged over 200 simulations. Recruitment parameter $H=1.4$ [$\eta=0.2$ in Eq. (17)] and population size $n=300$.

B_Σ	B_Δ	$\langle x\rangle$ (simulation)	Theoretical value
		Below threshold, $B_\Sigma<\tilde{B}_\Sigma$	
0.3	0.1	0.4990	$x^*=0.5$
		(± 0.03)	
		Above threshold, $B_\Sigma>\tilde{B}_\Sigma$	
0.5	0.2	0.8304	$\hat{x}_+=0.8370$
		(± 0.01)	

The evolution of the fraction x_t of the population at source A has been averaged over 200 different simulations of the Markov chain (2). The results show that, after 250 time steps, this average stabilizes around a constant value $\langle x\rangle$ within small residual fluctuations due to the finite population size, and provide an empirical proof that the system has reached equilibrium. For two such experiments, conducted below and above threshold (23), the values of $\langle x\rangle$ are compared in Table I against the theoretical values $x^*=0.5$ and $\hat{x}_+$ obtained by numerical solution of the fixed point equation. The results are seen to agree well.

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[1] Y. Kamp, Biol. Cybern. **77**, 447 (1997).
[2] D.J. Amit, *Modeling Brain Function* (Cambridge University Press, Cambridge, 1989).
[3] J.P. Provost and G. Vallee, Phys. Rev. Lett. **50**, 598 (1983).
[4] H.E. Daniels, Ann. Math. Stat. **25**, 631 (1954).
[5] N.G. De Bruijn, *Asymptotic Methods in Analysis* (North-Holland, Amsterdam, 1958).
[6] R.P. Fletcher, C. Cannings, and P. G. Blackwell, in *Advances in Artificial Life*, edited by F. Morán *et al.* (Springer, Berlin, 1995), p. 772.
[7] A. Kirman, Q. J. Econ. **108**, 137 (1993).